

# Linkage of Quadratic Pfister Forms

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## Abstract

A set of quadratic  $n$ -fold Pfister forms is linked if it has a common  $(n - 1)$ -fold factor. We study this notion over a field  $F$  of characteristic 2, where one has to distinguish between left- and right-linkage. Every linked set must be “tight”, namely generate a group of forms in  $I_q^n F / I_q^{n+1} F$ . Following Sivatzki, who studied triples of quaternion algebras, we associate to any tight set an invariant in  $I_q^{n+1} F$ , which is zero whenever the set is left-linked. In fact, a left-linked set generates a group of forms in  $I_q^n F$ ; we call such a set “strongly tight”.

We show that a right-linked set is strongly tight if and only if it is pairwise left-linked. For any  $n$ , we construct a set of  $n + 1$  quadratic  $n$ -fold Pfister forms which is strongly tight, and in particular has zero invariant, but does not have any common 1-fold Pfister factor. For  $n = 2$  this answers a problem of Sivatzki on the negative.

*Keywords:* Quadratic Forms, Pfister Forms, Linkage, Quaternion Algebras  
*2010 MSC:* 11E81 (primary); 11E04, 16K20 (secondary)

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*Preprint submitted to ??*

*November 12, 2015*

## 1. Introduction

Linkage of quaternion algebras was the subject of several papers in recent years. Two quaternion algebras are linked if they share isomorphic subfields; thus in characteristic 2 the linkage can be separable or non-separable. Draxl proved in [Dra83] that if two quaternion algebras are non-separably linked, then they are separably linked as well. [Lam02] simplified the proof and provided a counterexample for the converse statement. This result was generalized in [Chaar] to cyclic  $p$ -algebras of any prime degree. In [EV05] a necessary and sufficient condition was given for two quaternion algebras which are separably linked to be non-separably linked. Faivre, in [Fai06], proved analogous statements for quadratic Pfister forms.

The notion of linkage can be studied for more than two quaternion algebras. Recall that  ${}_2\text{Br}(F)$  is the exponent 2 subgroup of the Brauer group. Let  $G$  be a subgroup of  ${}_2\text{Br}(F)$ . We say that  $G$  is linked if there is a quadratic extension of  $F$  that splits every element of  $G$  (for example the subgroup  $\langle [Q], [Q'] \rangle$  is linked if and only if  $Q, Q'$  share a common subfield). We say that  $G$  is **tight** if every element of  $G$  is a quaternion algebra. We are mostly interested in tight sets, because being tight is a necessary condition for linkage.

**Remark 1.1.** *Let  $G \subseteq {}_2\text{Br}(F)$  be a subgroup. If  $G$  is tight then every two elements of  $G$  are linked (by Albert-Sah's theorem, [Alb72, Sah72]).*

Peyre gave in [Pey95] an example of three quaternion algebras  $Q_1, Q_2, Q_3$  (in characteristic not 2) such that  $G = \langle [Q_1], [Q_2], [Q_3] \rangle$  is tight but not linked. In [Siv14], Sivatski provided an invariant  $\Sigma'_G \in I_q^3 F / I_q^4 F$  for tight subgroups  $G = \langle [Q_1], [Q_2], [Q_3] \rangle$  which vanishes if  $G$  is linked. An example where the invariant vanishes but  $G$  is not linked was recently constructed in [QMT15].

In this paper we present a theory of linkage for Pfister forms. Quaternion algebras can be interpreted as 2-fold Pfister forms modulo  $I_q^3$ . We say that a set of  $n$ -fold Pfister forms is **linked** if there is an  $(n-1)$ -fold Pfister form which is a common factor to all the forms in the set. Generalizing the notion introduced above, we say that a set of  $n$ -fold Pfister forms is **tight** if every element of the group it generates in  $I_q^n F / I_q^{n+1} F$  is represented by an  $n$ -fold Pfister form. Every linked set is tight, and our goal is to propose further restrictions on a tight set, bringing it closer to linkage.

Let  $S$  be a tight set of  $n$ -fold Pfister forms, generating a group of order  $2^s$  in  $I_q^n F / I_q^{n+1} F$ . Motivated by [Siv14], we associate to  $S$  a quadratic form  $\Sigma_S \in I_q^{n+1} F$ , in a way that if  $S$  is linked, then  $\Sigma_S \in I_q^{n+s-1} F$ .

Over fields of characteristic 2, one has to distinguish between bilinear and quadratic linkage, which, following the standard notation for quadratic Pfister forms, we term left- and right-linkage, respectively (see Definition 3.1). We recall that Faivre proved in [Fai06] that a left-linked pair of forms is also right-linked. On the other hand, we show that a right-linked pair of  $n$ -fold Pfister forms is left-linked if and only if their invariant is zero (see Theorem 5.3).

Focusing mostly on fields of characteristic 2, where left-linkage and right-linkage are distinct properties (Definition 3.1), we recall that Faivre proved in [Fai06] that a left-linked pair of forms is also right linked. On the other hand, we show that a right-linked pair of  $n$ -fold Pfister forms is left-linked if and only if their invariant is zero (see Theorem 5.3).

The notion of linkage for larger sets is more complicated. While a right-linked set is always tight, a left-linked set must be **strongly tight**, namely it generates a subgroup whose elements are forms not only in the quotient  $I_q^n F / I_q^{n+1} F$ , but in  $I_q^n F$  itself. This notion is clearly special to characteristic 2. When this is the case, the invariant  $\Sigma_S$  is zero. In Section 7 we show that a tight set  $S$  is strongly tight if and only if the invariants associated to its subsets are all zero. This allows us to conclude that a right-linked set which is pairwise left-linked is in fact strongly tight.

Section 8 is devoted to quaternion algebras. We observe that the anisotropic part of a set of  $s$  2-fold Pfister forms has dimension at most  $2^{s+1}$ ; in particular if  $S$  is right-linked and contains some pair of left-linked forms, then  $\Sigma_S = 0$ . In Sections 9 and 10 we develop an additive notation for quadratic Pfister forms in characteristic 2, à la Jacobson, and provide a set of  $n$ -fold forms, of size  $n + 1$ , which is strongly tight but do not have any 1-fold common factor. This answers [Siv14, Open Question 1] in characteristic 2. We conclude the paper with an example of a tight triplet of quaternion algebras whose invariant is nonzero in  $I_q^3 F / I_q^4 F$ .

## 2. Quadratic Pfister Forms

We recall the basic structure of quadratic forms, including the case of characteristic 2, as described e.g. in [EKM08]. Let  $F$  be a field. If  $\text{char}(F) \neq 2$  we tacitly assume that  $F$  contains a square root of  $-1$ . The hyperbolic

plane  $\mathbb{H}$  is the unique (up to isometry) two-dimensional nonsingular isotropic quadratic form  $\varphi(u_1, u_2) = u_1 u_2$ . Every quadratic form  $\varphi$  over  $F$  decomposes uniquely as

$$\varphi \simeq i \times \mathbb{H} \perp \varphi_{\text{an}} \perp j \times \langle 0 \rangle$$

where  $\varphi_{\text{an}}$  is anisotropic and  $\langle 0 \rangle$  is the 1-dimensional zero form. We call  $i$  the Witt index of  $\varphi$  and denote it by  $i_W(\varphi)$ . The form is nonsingular if and only if  $j = 0$ .

The diagonal bilinear form  $b(u, v) = \alpha_1 u_1 v_1 + \cdots + \alpha_n u_n v_n$  is denoted by  $\langle \alpha_1, \dots, \alpha_n \rangle$ . The same notation stands for the quadratic form  $b(u, u)$ . The notation  $[\alpha, \beta]$  stands for the 2-dimensional quadratic form  $\alpha u_1^2 + u_1 u_2 + \beta u_2^2$ . An anisotropic quadratic form can be decomposed as an orthogonal sum

$$\langle \alpha_1, \dots, \alpha_t \rangle \perp [\beta_1, \gamma_1] \perp \cdots \perp [\beta_r, \gamma_r] \quad (1)$$

where  $\alpha_1, \dots, \alpha_t \neq 0$ ; in characteristic not 2 we may assume  $r = 0$  (indeed  $[\beta, \gamma] = \langle \beta, \gamma - (4\beta)^{-1} \rangle$ ).

A quadratic form is nondegenerate if it remains nonsingular under any field extension. In characteristic not 2, every nonsingular form is nondegenerate. In characteristic 2, a form  $i \times \mathbb{H} \perp \varphi_{\text{an}}$  is nondegenerate if and only if  $t \leq 1$  in the decomposition (1) of  $\varphi_{\text{an}}$ .

Two quadratic forms are Witt equivalent if their underlying anisotropic subforms are isometric. The group of Witt equivalence classes of even dimensional nondegenerate quadratic forms over  $F$ , with respect to orthogonal sum, is denoted by  $I_q^1 F$ . This is a module over the Witt ring  $WF$  (of symmetric nondegenerate bilinear forms, modulo hyperbolic forms), with respect to tensor product.

We abuse the terminology by often identifying a quadratic form with its Witt equivalence class. The group  $I_q^1 F$  is generated by the forms  $[\alpha, \beta]$ ; if  $\text{char}(F) \neq 2$ , it is also generated by the forms  $\langle \alpha, \beta \rangle$  ( $\alpha, \beta \in F^\times$ ). Since we assume  $\sqrt{-1} \in F$  when the characteristic is not 2,  $I_q^1 F$  is annihilated by the element  $\langle 1, 1 \rangle$  of the Witt ring.

The bilinear forms  $\langle \langle \beta \rangle \rangle = \langle 1, \beta \rangle$  are called (bilinear) 1-fold Pfister forms. These forms span the basic ideal  $IF$  of  $WF$ . Powers of  $IF$  are denoted  $I^n F$ . The tensor products  $\langle \langle \beta_1, \dots, \beta_n \rangle \rangle = \langle \langle \beta_1 \rangle \rangle \otimes \cdots \otimes \langle \langle \beta_n \rangle \rangle$  are called bilinear  $n$ -fold Pfister forms.

The quadratic form  $[1, \alpha]$  is called a (quadratic) 1-fold Pfister forms, and denoted by  $\langle \langle \alpha \rangle \rangle$  (in characteristic not 2 the customary notation is  $\langle \langle \alpha \rangle \rangle = \langle 1, \alpha \rangle$ , but  $\langle \langle \alpha \rangle \rangle = \langle \langle \alpha + 1/4 \rangle \rangle$ ). For any quadratic form  $\varphi$  and  $\beta_1, \dots, \beta_n \in$

$F^\times, \langle \beta_1, \dots, \beta_n \rangle \otimes \varphi = \beta_1 \varphi \perp \dots \perp \beta_n \varphi$ . For any integer  $n \geq 2$ , we define the quadratic  $n$ -fold Pfister form  $\langle \langle \beta_1, \dots, \beta_{n-1}, \alpha \rangle \rangle$  as  $\langle \langle \beta_1, \dots, \beta_{n-1} \rangle \rangle \otimes \langle \langle \alpha \rangle \rangle$ . A Pfister form is isotropic if and only if it is hyperbolic. We defined  $I_q^n F$  to be the subgroup of  $I_q^1 F$  generated by the scalar multiples of quadratic  $n$ -fold Pfister forms. It follows that  $I^m F \cdot I_q^k F = I_q^{m+k} F$  for every  $m, k \geq 0$ , where  $I^m F$  is the  $m$ th power of the ideal  $IF$ .

### 3. Linkage and tightness

If  $b$  is a bilinear  $m$ -fold Pfister form and  $\varphi$  is a quadratic  $k$ -fold Pfister form, then  $b \otimes \varphi$  is a quadratic  $(m+k)$ -fold Pfister form. In this case we say that  $b$  is a left divisor and  $\varphi$  is a right divisor of  $b \otimes \varphi$ . A set  $S$  of quadratic  $(m+k)$ -fold Pfister forms is said to be  $k$ -right-linked if there is a quadratic  $k$ -fold Pfister form  $\varphi$  which is a right divisor of every element of  $S$ . Similarly,  $S$  is said to be  $m$ -left-linked if there is a bilinear  $m$ -fold Pfister form  $b$  which is a left divisor of every element of  $S$ .

Throughout, we are only interested in linkage where the free part is a (bilinear or quadratic) 1-fold Pfister form:

**Definition 3.1.** *A set of quadratic  $n$ -fold Pfister forms is **right-linked** if the set is  $(n-1)$ -right-linked and **left-linked** if they are  $(n-1)$ -left-linked.*

If  $\text{char}(F) \neq 2$  there is a one-to-one correspondence between quadratic and symmetric bilinear forms, so the notions of left- and right- linkage coincide.

We recall a necessary condition for a set of forms to be linked:

**Remark 3.2** ([EKM08, Cor. 23.9]). *If two  $n$ -fold Pfister forms are equivalent modulo  $I_q^{n+1} F$  then they are isometric.*

**Definition 3.3.** *We say that a set of quadratic  $n$ -fold Pfister forms is **tight** if every element of the group it generates in  $I_q^n F / I_q^{n+1} F$  is represented by a Pfister form (which is unique, by Remark 3.2). The set is **strongly tight** if the group it generates in  $I_q^n F$  consists of Pfister forms.*

Obviously, a strongly tight set is tight. A subset of a (strongly) tight set is itself (strongly) tight. In the case of quaternions (namely  $n = 2$ ), a group generated by a tight set is called in [QMT15] a ‘quaternionic subgroup’ (we could call a group generated by a tight set a ‘symbolic subgroup’).

By [EKM08, Proposition 24.5], every two forms in a tight set are right-linked. We move on to show that every right-linked set is tight, and every left-linked set is strongly tight.

**Proposition 3.4.** *Let  $S$  be a right-linked set of  $n$ -fold Pfister forms. Then there are a quadratic  $(n-1)$ -fold Pfister form  $\varphi$  and elements  $\beta_1, \dots, \beta_s \in F^\times$  such that the elements of the group generated by  $S$  in  $I_q^n F / I_q^{n+1} F$  are represented by the Pfister forms  $\langle \beta_{i_1} \cdots \beta_{i_k} \rangle \otimes \varphi$  for the subsets  $\{i_1, \dots, i_k\} \subseteq \{1, \dots, k\}$ .*

*Proof.* By the assumption, the elements of  $S$  are of the form  $\langle \beta_i \rangle \otimes \varphi$  where  $\varphi$  is a fixed  $(n-1)$ -fold Pfister form. Since  $\langle \beta \rangle \perp \langle \beta' \rangle \sim \langle \beta\beta' \rangle \perp \langle \beta, \beta' \rangle \equiv \langle \beta\beta' \rangle \pmod{I_q^2 F}$ , the forms  $\langle \beta_{i_1} \cdots \beta_{i_k} \rangle \otimes \varphi$  represent the elements in the generated subgroup.  $\square$

In particular,

**Corollary 3.5.** *A right-linked set is tight.*

### 3.1. Left-linkage in characteristic 2

Now consider left-linkage and assume  $F$  has characteristic 2.

**Proposition 3.6.** *In characteristic 2, a left-linked set of quadratic  $n$ -fold Pfister forms is strongly tight.*

*Moreover if  $S$  is left-linked then there are a bilinear  $(n-1)$ -fold Pfister form  $b$  and elements  $\alpha_1, \dots, \alpha_s \in F$  such that the elements of the group generated by  $S$  in  $I_q^n F$  are the Pfister forms  $b \otimes \langle \beta_{i_1} + \cdots + \beta_{i_k} \rangle$  for the subsets  $\{i_1, \dots, i_k\} \subseteq \{1, \dots, k\}$ .*

*Proof.* Same as the proof of Proposition 3.4, using the fact that in characteristic 2,  $\langle \beta \rangle + \langle \beta' \rangle \sim \langle \beta_1 + \beta_2 \rangle$  is an equality in  $I_q F$ , and not just modulo  $I_q^2 F$  [EKM08, Example 7.23].  $\square$

In fact we can prove a stronger statement. We say that a tight set  $S$  of  $n$ -fold Pfister forms is **completely pairwise left-linked** if the representing forms of the group generated by  $S$  in  $I_q^n F / I_q^{n+1} F$  are pairwise left-linked. (Obviously a left-linked set is completely pairwise left-linked).

**Corollary 3.7.** *A completely pairwise left-linked set  $S$  of  $n$ -fold Pfister forms is strongly tight.*

*Proof.* Let  $G$  denote the group generated by  $S$  in  $I_q^n F / I_q^{n+1} F$ . Since  $S$  is tight, for every  $g \in G$  there is a (unique) quadratic  $n$ -fold Pfister form  $\varphi_g \in g + I_q^{n+1} F$ . Let  $g, g' \in G$  be distinct elements; then  $\varphi_g$  and  $\varphi_{g'}$  are left-linked, so by Proposition 3.6 the pair  $\varphi_g, \varphi_{g'}$  is strongly tight, and so  $\varphi_{g+g'} = \varphi_g + \varphi_{g'}$  in  $I_q^n F$ . Since  $\varphi_g$  have order 2 in  $I_q^n F$ , we proved that  $\{\varphi_g : g \in G\}$  is a subgroup of  $I_q^n F$ .  $\square$

#### 4. An obstruction for strong tightness

Given a tight set  $S$  of  $n$ -fold Pfister forms, we now define an invariant  $\Sigma_S$  which vanishes if the set is strongly tight (and thus if the set is left-linked in characteristic 2), and is a higher Pfister form if the set is right-linked. This is essentially the invariant defined by Sivatzki for triples of quaternion algebras, which takes values in  $I_q^3 F / I_q^4 F$  (more details are given in Section 8 below).

**Definition 4.1.** *Let  $S$  be a tight set of (more than one)  $n$ -fold Pfister forms. We define  $\Sigma_S$  to be the sum in  $I_q^n F$  of the unique representatives of the elements of the subgroup generated by  $S$  in  $I_q^n F / I_q^{n+1} F$  (which exist by assumption and are unique by Remark 3.2, so  $\Sigma_S$  is well defined).*

**Proposition 4.2.** 1. *For any tight set  $S$  of  $n$ -fold Pfister forms,  $\Sigma_S \in I_q^{n+1} F$ .*  
 2. *If  $S$  is strongly tight then  $\Sigma_S = 0$ .*

*Proof.* The image of  $\Sigma_S$  in  $I_q^n F / I_q^{n+1} F$  is the sum of all the elements in a noncyclic group of exponent 2, which is zero; this proves the first claim. The second claim follows from the same argument, applied to  $I_q^n F$  rather than  $I_q^n F / I_q^{n+1} F$ .  $\square$

**Proposition 4.3.** *Let  $S$  be a right-linked set of  $n$ -fold Pfister forms, generating a group of order  $2^s$  modulo  $I_q^{n+1} F$ . Then  $\Sigma_S$  is an  $(n + s - 1)$ -fold Pfister form.*

*Proof.* Let  $\varphi$  be the common right divisor. By Remark 3.4, the elements of the group generated by  $S$  are represented by the forms  $\langle\langle\beta_{i_1} \cdots \beta_{i_k}\rangle\rangle \otimes \varphi$ , where  $\{i_1, \dots, i_k\} \subseteq \{1, \dots, s\}$ .

One proves by induction on  $t$  that for every  $\alpha, \beta_1, \dots, \beta_t \in F^\times$ ,

$$\sum_{\{i_1, \dots, i_k\} \subseteq \{1, \dots, t\}} \langle\langle\alpha \beta_{i_1} \cdots \beta_{i_k}\rangle\rangle = \langle\langle\beta_1, \dots, \beta_t\rangle\rangle + \langle\langle\alpha, \beta_1, \dots, \beta_t\rangle\rangle$$

in  $I_q F$ . Taking  $\alpha = 1$ , the sum of the left 1-fold factors of the representatives in the group generated by  $S$  is seen to be the  $s$ -fold form  $\langle\langle\beta_1, \dots, \beta_s\rangle\rangle$ , so  $\Sigma_S = \langle\langle\beta_1, \dots, \beta_s\rangle\rangle \otimes \varphi$ .  $\square$

## 5. Left- and Right- Linkage of pairs

Throughout this section we assume  $\text{char}(F) = 2$ . We start by recalling some results from [Fai06]:

**Corollary 5.1** ([Fai06, Corollary 2.1.4]). *If two Pfister forms are left-linked then they are also right-linked.*

However, there are pairs of right-linked forms which are not left-linked, [Fai06, Proposition 2.5.1].

If  $b$  is a bilinear Pfister form, the pure subform is the unique form  $b'$  such that  $b = \langle 1 \rangle \perp b'$ . For a quadratic Pfister form  $\varphi = b \otimes \langle\langle\beta, \alpha\rangle\rangle$ , the pure subform is defined as the quadratic form, of dimension  $2^n - 1$ ,

$$\varphi' = b' \otimes \langle\langle\alpha\rangle\rangle \perp b \otimes \langle\beta\rangle \otimes \langle\langle\alpha\rangle\rangle \perp \langle 1 \rangle.$$

**Theorem 5.2** ([Fai06, Theorem 2.5.5]). *Two  $n$ -fold Pfister forms  $\phi$  and  $\psi$  are left-linked if and only if  $i_W(\phi' \perp \psi') \geq 2^{n-1} - 1$ .*

Using this criterion of Faivre for left-linkage of  $n$ -fold Pfister forms, we can give a criterion for left-linkage of right-linked forms:

**Theorem 5.3.** *Two right-linked  $n$ -fold Pfister forms*

$$\varphi = \langle\langle\beta, \alpha_1, \dots, \alpha_{n-1}\rangle\rangle, \quad \psi = \langle\langle\gamma, \alpha_1, \dots, \alpha_{n-1}\rangle\rangle$$

*are left-linked if and only if the  $(n+1)$ -fold Pfister form  $\langle\langle\gamma, \beta, \alpha_1, \dots, \alpha_{n-1}\rangle\rangle$  is hyperbolic.*

*Proof.* Write  $b = \langle\langle\alpha_1, \dots, \alpha_{n-2}\rangle\rangle$ , so that  $\varphi = b \otimes \langle\langle\beta, \alpha_{n-1}\rangle\rangle$  and  $\psi = b \otimes \langle\langle\gamma, \alpha_{n-1}\rangle\rangle$ . By definition

$$\varphi' = b' \otimes \langle\langle\alpha_{n-1}\rangle\rangle \perp b \otimes \langle\beta\rangle \otimes \langle\langle\alpha_{n-1}\rangle\rangle \perp \langle 1 \rangle$$

and

$$\psi' = b' \otimes \langle\langle\alpha_{n-1}\rangle\rangle \perp b \otimes \langle\gamma\rangle \otimes \langle\langle\alpha_{n-1}\rangle\rangle \perp \langle 1 \rangle;$$



since  $\langle 1 \rangle \perp \langle 1 \rangle \cong \langle 1 \rangle \perp \langle 0 \rangle$ , we have that

$$\varphi' \perp \psi' = (2^{n-1} - 2) \times \mathbb{H} \perp \langle \beta, \gamma \rangle \otimes b \otimes \langle \alpha_{n-1} \rangle \perp \langle 1, 0 \rangle,$$

and so the Witt index  $i_W(\varphi' \perp \psi')$  is equal to  $2^{n-1} - 2$ , plus the Witt index of the form  $\theta = \langle \beta, \gamma \rangle \otimes b \otimes \langle \alpha_{n-1} \rangle \perp \langle 1 \rangle$ . By Faivre's criterion,  $\varphi$  and  $\psi$  are left-linked if and only if  $\mathbb{H} \subseteq \theta$ .

Since  $\langle 1 \rangle \subseteq b \otimes \langle \alpha_{n-1} \rangle$ , we have that  $\theta = \langle 1 \rangle \perp \langle \beta, \gamma \rangle \otimes b \otimes \langle \alpha_{n-1} \rangle \subseteq \langle 1, \beta, \gamma, \beta\gamma \rangle \otimes b \otimes \langle \alpha_{n-1} \rangle$ , so if  $\mathbb{H} \subseteq \theta$ , the Pfister form  $\langle \langle \beta, \gamma \rangle \rangle \otimes b \otimes \langle \alpha_{n-1} \rangle$  is isotropic and thus hyperbolic.

On the other hand if  $\langle \langle \beta, \gamma \rangle \rangle \otimes b \otimes \langle \alpha_{n-1} \rangle$  is hyperbolic, then  $\langle \beta, \gamma \rangle \otimes b \otimes \langle \alpha_{n-1} \rangle \simeq \langle 1, \beta\gamma \rangle \otimes b \otimes \langle \alpha_{n-1} \rangle$ . Adding  $\langle 1 \rangle$  to both forms, we get  $\theta \simeq \langle 1, \beta\gamma \rangle \otimes b \otimes \langle \alpha_{n-1} \rangle \perp \langle 1 \rangle$  which is obviously isotropic.  $\square$

**Example 5.4.** *The 2-fold Pfister forms  $\langle \langle \beta, \alpha \rangle \rangle$  and  $\langle \langle \beta', \alpha \rangle \rangle$  are left-linked if and only if the 3-fold Pfister form  $\langle \langle \beta, \beta', \alpha \rangle \rangle$  is hyperbolic.*

**Corollary 5.5.** *Suppose  $I_q^{n+1}F = 0$ . Then two  $n$ -fold Pfister forms over  $F$  are right-linked if and only if they are left-linked.*

Fix  $s \geq 3$ . We give an example of a right-linked set  $S$  (of size  $s$ ), which is not pairwise left-linked, despite the fact that  $\Sigma_S = 0$ :

**Example 5.6.** *Let  $F = k(\beta_1, \dots, \beta_s, \alpha_1, \dots, \alpha_{n-1})$  be the function field in  $n + s - 1$  algebraically independent variables over a field  $k$  of characteristic 2. Let  $S$  be the set of the  $n$ -fold Pfister forms  $\langle \langle \beta_i, \alpha_1, \dots, \alpha_{n-1} \rangle \rangle$  where  $i = 1, \dots, s$ ; it is right-linked, and therefore tight. By Proposition 4.3,  $\Sigma_S = \langle \langle \beta_1, \dots, \beta_s, \alpha_1, \dots, \alpha_n \rangle \rangle$  is the generic  $(n + s)$ -fold Pfister form.*

*Let  $K$  be the function field of the form  $\Sigma_S$ . Consider any two forms  $\langle \langle \beta, \alpha_1, \dots, \alpha_{n-1} \rangle \rangle$  and  $\langle \langle \beta', \alpha_1, \dots, \alpha_{n-1} \rangle \rangle$ , where  $\beta$  and  $\beta'$  are distinct products of subsets of the  $\beta_i$ . The form  $\langle \langle \beta, \beta', \alpha_1, \dots, \alpha_{n-1} \rangle \rangle$  is not split by  $K$  (see [HL06, Theorem 1.1]), so by Theorem 5.3, the two forms are not left-linked over  $K$ .*

*Therefore, by restricting everything to  $K$ , we obtain a right-linked set of forms with  $\Sigma = 0$ , where no two forms are left-linked.*

## 6. Tightness and strong tightness of pairs

In this section we assume  $S = \{\varphi_1, \varphi_2\}$  is a set of two quadratic  $n$ -fold Pfister forms, and compare the notions of left- and right- linkage with tightness and strong tightness.

**Proposition 6.1.** *The pair  $\varphi_1, \varphi_2$  is right-linked if and only if it is tight.*

*Proof.* This is Corollary 3.5 and [EKM08, Prop. 24.5] which was quoted earlier.  $\square$

**Proposition 6.2.** *Let  $S = \{\varphi_1, \varphi_2\}$  and assume  $F$  has characteristic 2. The following are equivalent:*

1.  *$S$  is left-linked.*
2.  *$S$  is strongly tight.*
3.  *$S$  is tight and  $\Sigma_S = 0$ .*

*Proof.* A left-linked set is strongly tight by Proposition 3.6. A strongly tight set  $S$  is tight and has  $\Sigma_S = 0$  by Proposition 4.2.2. It remains to show that if  $S = \{\varphi_1, \varphi_2\}$  is a tight pair and  $\Sigma_S = 0$  then  $S$  is left-linked. By Proposition 6.1,  $S$  is right-linked, so we can write them as  $\varphi_i = \langle\langle\alpha_i\rangle\rangle \otimes \psi$  for a quadratic Pfister form  $\psi$ . The invariant  $\Sigma_S$  was computed in Proposition 4.3 to be  $\Sigma_S = \langle\langle\alpha_1, \alpha_2\rangle\rangle \otimes \psi$ . By Theorem 5.3,  $S$  is left-linked.  $\square$

## 7. From tightness to strong tightness through invariants

The notions of tightness and strong tightness, defined above for sets of quadratic Pfister forms, can be easily generalized and treated in a general setting. The application to Pfister forms is transparent.

Let  $V$  be a vector space over the field of two elements,  $U \subseteq V$  a subspace, and  $P \subseteq V$  a designated spanning set, such that every coset of  $V/U$  contains at most one element of  $P$  (in our context  $P$  is the set of quadratic  $n$ -fold Pfister forms,  $V = I_q^n F$  and  $U = I_q^{n+1} F$ ). A set  $S \subseteq V$  is **tight** if the generated subspace  $\langle S \rangle$  is contained in  $P + U$ ; and **strongly tight** if  $\langle S \rangle \subseteq P$ . For a tight set  $S$ , we define  $\Sigma_S$  to be the sum  $\sum p_{S'} \in V$ , where  $p_{S'} \in P$  are the unique elements such that  $\{p_{S'} + U\}$  is the group generated by  $S$  in  $V/U$ , and  $S'$  ranges over the subsets of  $S$ . As long as  $|\langle S \rangle| > 2$ , we have that  $\Sigma_S \in U$ .

Let us say that a set is **almost strongly tight** if it is tight and every proper subset is strongly tight. A strongly tight set is almost strongly tight.

**Proposition 7.1.** *Let  $S = \{\varphi_1, \dots, \varphi_s\} \subseteq P$  be an almost strongly tight set of cardinality  $s > 1$ . Then  $S$  is strongly tight if and only if  $\Sigma_S = 0$ .*

*Proof.* We may assume  $S$  is linearly independent, because otherwise the claim holds by assumption. For every  $S' \subseteq S$ , let  $p_{S'}$  be the unique representative from  $P$  of the element  $\sum_{i \in S'} \varphi_i + U$ . If  $S' \subsetneq S$  then  $p_{S'} = \sum_{i \in S'} \varphi_i$  because  $S'$  is strongly tight. By definition

$$\begin{aligned} \Sigma_S &= \sum_{S' \subseteq S} p_{S'} = \sum_{S' \subsetneq S} p_{S'} + p_S = \sum_{S' \subsetneq S} \sum_{i \in S'} \varphi_i + p_S \\ &= \sum_{S' \subseteq S} \sum_{i \in S'} \varphi_i + (p_S - \sum_{i \in S} \varphi_i) = p_S - \sum_{i \in S} \varphi_i \end{aligned}$$

because the sum of elements of a noncyclic group of exponent 2 is zero. So  $\sum_{i \in S} \varphi_i = p_S - \Sigma_S$ , which is in  $P$  if and only if  $\Sigma_S = 0$ .  $\square$

**Theorem 7.2.** *A tight set  $S = \{\varphi_1, \dots, \varphi_s\} \subseteq P$  is strongly tight if and only if  $\Sigma_{S'} = 0$  for every subset  $S' \subseteq S$  of cardinality  $> 1$ .*

*Proof.* If  $S$  is strongly tight then for every subset  $S' \subseteq S$  we have that  $\langle S' \rangle \subseteq P$ , and the sum over a non-cyclic group of exponent 2 is zero.

For the other direction, we prove that every subset of size  $2 \leq k \leq |S|$  is strongly tight, by induction on  $k$ . Sets of size 1 are strongly tight by definition, so we may assume every subset of size  $k - 1$  is strongly tight. Let  $S' \subseteq S$  be a subset of size  $k$ . By Proposition 7.1,  $S'$  is strongly tight because  $\Sigma_{S'} = 0$  by assumption.  $\square$

### 7.1. Applications to quadratic forms

Consider a set  $S = \{\varphi_1, \dots, \varphi_s\}$  of  $n$ -fold Pfister forms. For  $S$  to be left-linked, it is necessary that  $S$  is strongly tight; and for that, it is necessary that  $S$  is almost strongly tight.

**Corollary 7.3.** *Let  $S = \{\varphi_1, \dots, \varphi_s\}$  be an almost strongly tight set of  $n$ -fold Pfister forms. Then  $S$  is strongly tight if and only if  $\Sigma_S = 0$ . (This is Proposition 7.1).*

For a set  $S$  to be left-linked, it must be pairwise left-linked, and it must be strongly tight. It turns out that if  $S$  is right-linked, then the two conditions are equivalent:

**Theorem 7.4.** *Let  $S$  be a right-linked set of  $n$ -fold Pfister forms. The following are equivalent when  $F$  has characteristic 2:*

1.  $S$  is pairwise left-linked.

2.  $S$  is completely pairwise left-linked.
3.  $S$  is strongly tight.

*Proof.* (2)  $\implies$  (1) is trivial. (3)  $\implies$  (2): if  $S$  is strongly tight, then every pair of elements is left-linked by Proposition 6.2. (1)  $\implies$  (3): By assumption the elements of  $S$  are  $\langle\langle\beta_i\rangle\rangle\otimes\varphi$ ,  $i = 1, \dots, s$ , where  $\beta_1, \dots, \beta_s \in F^\times$  and  $\varphi$  is a quadratic  $(n-1)$ -fold Pfister. By Theorem 5.3, the forms  $\langle\langle\beta_i, \beta_j\rangle\rangle\otimes\varphi$  are hyperbolic for every  $i \neq j$ . Moreover,  $S$  is tight. By Proposition 3.4, if  $S' \subseteq S$  is a subset of size  $> 1$ , then  $\Sigma_{S'} = \langle\langle\beta_{i_1}, \dots, \beta_{i_k}\rangle\rangle\otimes\varphi$  for a suitable set  $\{i_1, \dots, i_k\} \subseteq \{1, \dots, s\}$ . Every invariant of this form is divisible by one of the  $\langle\langle\beta_i, \beta_j\rangle\rangle\otimes\varphi$ , and is thus hyperbolic. By Theorem 7.2,  $S$  is strongly tight.  $\square$

We do not know if a right-linked set which is pairwise left-linked, is left-linked.

**Corollary 7.5.** *Let  $\alpha_1, \dots, \alpha_s \in F^\times$  ( $F$  has characteristic 2) and let  $\psi$  be a quadratic  $(n-1)$ -fold Pfister form. Assume that the forms  $\langle\langle\alpha_i\rangle\rangle\otimes\psi$  are pairwise left-linked. Then*

$$\langle\langle\alpha_1\rangle\rangle \perp \dots \perp \langle\langle\alpha_s\rangle\rangle \perp \langle\langle\alpha_1 \cdots \alpha_s\rangle\rangle$$

*annihilates  $\psi$  in  $I_q^n F$ .*

*Proof.* The set of  $n$ -fold Pfister forms  $\langle\langle\alpha_i\rangle\rangle\otimes\psi$  is right-linked and pairwise left-linked, by assumption. By Theorem 7.4, this set is strongly tight and therefore the sum of the forms is the unique Pfister form representing the sum modulo  $I_q^{n+1}(F)$ , which is  $\langle\langle\alpha_1 \cdots \alpha_s\rangle\rangle\otimes\psi$ .  $\square$

**Corollary 7.6.** *Let  $[\beta, \alpha_i]$  be quaternion algebras. If for every  $i, j$  the quaternion algebras  $[\beta, \alpha_i]$  and  $[\beta, \alpha_j]$  share a nonseparable quadratic subfield, then*

$$\langle\langle\alpha_1, \beta]\rangle \perp \dots \perp \langle\langle\alpha_s, \beta]\rangle \sim \langle\langle\alpha_1 \cdots \alpha_s, \beta]\rangle$$

*in  $I_q^2 F$ . (This is Corollary 7.5 for  $n = 2$ , using Proposition 8.1).*

## 8. Quaternion Algebras

Two quaternion algebras are linked if they share a common subfield. The norm form of a quaternion algebra

$$[\alpha, \beta] = F\langle x, y \mid x^2 + x = \alpha, y^2 = \beta, yxy^{-1} = x + 1 \rangle \quad (2)$$

is  $\langle\langle\beta, \alpha\rangle\rangle$ , so left linkage translates to a common nonseparable subfield (of the form  $F[y]$ ), and right linkage to a common separable subfield (of the form  $F[x]$ ). The results of [Dra83], [Lam02] and [EV05] on the linkage of pairs of quaternion algebras were extended in [Fai06] to  $n$ -fold Pfister forms.

Sivatzsi studies [Siv14] triples of quaternion algebras over a field of characteristic not 2. If  $S = \{A, B, C\}$  is (in our terminology) tight, he defines an invariant  $\Sigma'_S \in I_q^3 F / I_q^4 F$  (which is the image of our invariant  $\Sigma_S \in I_q^3 F$ ). As pointed out by Sivatski,  $\Sigma'_S = 0$  is a necessary condition for  $A, B, C$  to be linked. Recall that in characteristic not 2, the symbol notation for quaternion algebras is

$$(\alpha, \beta)_{2,F} = F\langle x, y \mid x^2 = \alpha, y^2 = \beta, yxy^{-1} = -x \rangle.$$

Sivatski shows that the algebras  $(a, b)$ ,  $(b, c)$  and  $(c, a)$  over the function field  $k(a, b, c)$  are tight (and thus pair-wise linked), but since in this case  $\Sigma'_S = \langle\langle a, b, c \rangle\rangle$ , the algebras cannot be linked as a triplet. He then asks for an example with  $\Sigma'_S = 0$  which is not linked. Such an example was given in [QMT15].

### 8.1. Pair of quaternion algebras

Let  $Q_1, Q_2$  be quaternion algebras over  $F$ , of characteristic 2, and let  $\varphi_1, \varphi_2$  be the respective norm forms, which are 2-fold Pfister forms. Write  $Q_i = [\alpha_i, \beta_i]$ , so that  $\varphi_i = \langle\langle\beta_i, \alpha_i\rangle\rangle$ . We provide explicit conditions for the existence of left- or right-linkage.

**Proposition 8.1.** *The following conditions are equivalent.*

1.  $Q_1, Q_2$  have a common nonseparable subfield.
2.  $\varphi_1, \varphi_2$  are left-linked.
3. The form  $\mathbf{a}' = \beta_1[1, \alpha_1] \perp \beta_2[1, \alpha_2] \perp \langle 1 \rangle$  is isotropic.

*Proof.* The equivalence of the first two conditions is trivial. Taking  $n = 2$  in Theorem 5.2, since  $2^{n-1} - 1 = 1$ , the 2-fold Pfister forms  $\varphi_1$  and  $\varphi_2$  are left-linked if and only if  $\varphi'_1 \perp \varphi'_2$  is isotropic, but  $\varphi'_1 \perp \varphi'_2 = \beta_1[1, \alpha_1] \perp \beta_2[1, \alpha_2] \perp \langle 1, 1 \rangle \cong \beta_1[1, \alpha_1] \perp \beta_2[1, \alpha_2] \perp \langle 1, 0 \rangle$  as in the proof of Theorem 5.3.  $\square$

**Proposition 8.2.** *The following conditions are equivalent.*

1.  $Q_1, Q_2$  have a common separable subfield.

2.  $\varphi_1, \varphi_2$  are right-linked.
3. The form  $\mathfrak{a} = \beta_1[1, \alpha_1] \perp \beta_2[1, \alpha_2] \perp [1, \alpha_1 + \alpha_2]$  is isotropic.

*Proof.* Again the equivalence (1)  $\Leftrightarrow$  (2) is trivial. The form in (3) is the Albert form of  $Q_1 \otimes Q_2$ , which is isotropic if and only if  $Q_1, Q_2$  have a common subfield, see [KMRT98], Theorem 16.5 and Example 16.4.  $\square$

Notice that  $\mathfrak{a}' \subseteq \mathfrak{a}$ , which proves once more that left-linkage implies right-linkage. We also note that if  $\varphi_1, \varphi_2$  are right-linked, then we may assume  $\alpha_1 = \alpha_2 = \alpha$ , and the non-trivial part  $\beta_1[1, \alpha_1] \perp \beta_2[1, \alpha_2]$  of the Albert form  $\mathfrak{a}$  is similar to the Pfister form  $\mathfrak{a}_0 = \langle \langle \beta_1 \beta_2, \alpha \rangle \rangle$ , which is a factor of  $\Sigma_{\{\varphi_1, \varphi_2\}} = \langle \langle \beta_1, \beta_2, \alpha \rangle \rangle$ . Now Proposition 8.1 and Theorem 5.3 give two conditions for  $\varphi_1$  and  $\varphi_2$  to be left-linked, namely that  $\beta_1 \mathfrak{a}_0 \perp \langle 1 \rangle$  is isotropic and that  $\langle \langle \beta_1, \beta_2, \alpha \rangle \rangle$  is isotropic, respectively. Indeed, the two conditions are equivalent.

**Lemma 8.3.** *If  $\varphi_1, \varphi_2$  are left-linked over the function field  $F(\mathfrak{a})$ , then they are right-linked over  $F$ .*

*Proof.* Otherwise,  $\mathfrak{a}$  is anisotropic over  $F$  by Proposition 8.2, and the proper subform  $\mathfrak{a}'$  will remain anisotropic over  $F(\mathfrak{a})$ , contrary to Proposition 8.1.  $\square$

## 8.2. $\Sigma$ for 2-fold Pfister forms

Let  $S = \{\varphi_1, \dots, \varphi_s\}$  be a tight set of quadratic 2-fold Pfister forms over the field  $F$  of characteristic 2. Denote the corresponding quaternion algebras by  $Q_1, \dots, Q_s$ . By throwing out elements, we may assume the group generated by  $S$  in  $I_q^2 F / I_q^3 F \cong {}_2\text{Br}(F)$  has order  $2^s$ . For  $S' \subseteq S$ , let  $Q_{S'}$  denote the quaternion algebra similar to  $\bigotimes_{i \in S'} Q_i$  (which exists by tightness). Let  $\varphi_{S'}$  denote the associated norm form, so that  $\Sigma = \sum_{S' \subseteq S} \varphi_{S'}$  in the Witt module. Each of the (non-trivial) summands is a 2-fold Pfister form, so  $\dim(\Sigma) = 4(2^s - 1)$ .

**Proposition 8.4.**  $\dim(\Sigma_{\text{an}}) \leq 2^{s+1}$ .

*Proof.* Indeed, each of the  $2^s - 1$  summands has a subform of the form  $[1, *]$ , and  $[1, t] \perp [1, t'] \sim [1, t + t']$ , we obtain  $\dim(\Sigma_{\text{an}}) \leq 2(2^s - 1) + 2 = 2^{s+1}$ .  $\square$

**Proposition 8.5.** *Let  $S = \{\varphi_1, \dots, \varphi_s\}$  be a tight set. Assume that some pair of forms in the group generated by  $S$  is left-linked. Then  $\dim(\Sigma_{\text{an}}) < 2^{s+1}$ .*

*Proof.* By assumption there are distinct nonempty subsets  $S', S'' \subseteq S$  such that  $\varphi_{S'}$  and  $\varphi_{S''}$  are left-linked. Obviously  $Q_{S' \Delta S''} \sim Q_{S'} \otimes Q_{S''}$  (where  $\Delta$  is the symmetric difference), but moreover  $\varphi_{S' \Delta S''} = \varphi_{S'} + \varphi_{S''}$  in the Witt module by Proposition 6.2. Canceling these three summands,  $\Sigma$  is a sum of  $2^s - 4$  Pfister forms, so by the argument above,  $\dim(\Sigma_{\text{an}}) \leq 2(2^s - 4) + 2 = 2^{s+1} - 6 < 2^{s+1}$ .  $\square$

Theorem 7.4 shows that if  $S$  is right-linked and pairwise left-linked, then it is strongly tight, so in particular  $\Sigma = 0$ . We can prove more:

**Corollary 8.6.** *Assume  $S = \{\varphi_1, \dots, \varphi_s\}$  is right-linked, and that some pair of forms in the group generated by  $S$  is left-linked. Then  $\Sigma_S = 0$ .*

*Proof.* By Proposition 8.5  $\dim(\Sigma_{\text{an}}) < 2^{s+1}$ , but by Proposition 4.3,  $\Sigma$  is an  $(s+1)$ -fold Pfister form, so it is hyperbolic.  $\square$

## 9. Strong tightness does not imply linkage

Fix  $n \geq 1$ , and let  $k$  be any field of characteristic 2. Let  $E = k(\alpha_0, \dots, \alpha_n)$  be the transcendental extension of  $k$ .

**Theorem 9.1.** *There is a strongly tight set of  $n+1$  quadratic  $n$ -fold Pfister forms over  $E$  with no 1-fold common factor (in particular the set is neither left- nor right-linked).*

Taking  $n = 2$  and phrasing the result in terms of quaternion algebras, we conclude:

**Corollary 9.2.** *There is a triplet of quaternion division algebras  $A, B, C$  over  $k(\alpha_0, \alpha_1, \alpha_2)$ , such that every two algebras in  $G$  have a common nonseparable subfield, but there is no common subfield of  $A, B, C$ , in spite of the fact that the norm forms generate a group composed of 2-fold Pfister forms in  $I_q^2$  (in particular  $A, B, C$  generate a quaternionic group  $G$ ).*

After developing an additive notation for quadratic Pfister forms, we present the forms  $\psi_i$  in (3) below, and prove that  $S = \{\psi_0, \dots, \psi_n\}$  is strongly tight. The proof that the  $\psi_i$  are not linked is given in Section 10 using valuations.

### 9.1. Additive notation for quadratic Pfister forms

Following Jacobson's additive and symmetric notation for quaternion algebras in characteristic 2, we denote

$$((\alpha, \beta)) = \langle \langle \alpha, \alpha\beta \rangle \rangle$$

where  $\alpha \in F^\times$  and  $\beta \in F$ . These symbols obviously generate  $I_q^2 F$  ([Ara06] gives a presentation of  $I_q^2 F$  in these terms). More generally, we denote

$$((\alpha_1, \dots, \alpha_{n-1}, \beta)) = \langle \langle \alpha_1, \dots, \alpha_{n-1}, \alpha_1 \cdots \alpha_{n-1} \beta \rangle \rangle,$$

where  $\alpha_1, \dots, \alpha_{n-1} \in F^\times$  and  $\beta \in F$ . Clearly, every quadratic  $n$ -fold Pfister form has this presentation.

We begin with some basic properties of the symbols for  $n = 2$ :

**Proposition 9.3.** *The symbol  $((\cdot, \cdot))$  is a biadditive alternating form in  $I_q^2$ .*

*Proof.* Additivity in the right slot follows from

$$\begin{aligned} ((\alpha, \beta)) + ((\alpha, \beta')) &= \langle \langle \alpha, \alpha\beta \rangle \rangle + \langle \langle \alpha, \alpha\beta' \rangle \rangle \\ &= \langle \langle \alpha, \alpha(\beta + \beta') \rangle \rangle \\ &= ((\alpha, \beta + \beta')). \end{aligned}$$

The symbol is symmetric because  $((\beta, \alpha)) - ((\alpha, \beta)) = \langle \langle \beta, \alpha\beta \rangle \rangle - \langle \langle \alpha, \alpha\beta \rangle \rangle = \langle \beta, \alpha \rangle \langle \langle \alpha\beta \rangle \rangle = \langle \beta, \alpha \rangle [1, \alpha\beta] = [\beta, \alpha] \perp [\alpha, \beta] = 0$ ; this shows additivity in the left slot as well. Finally,  $((\alpha, \alpha)) = \langle \langle \alpha, \alpha^2 \rangle \rangle = \langle 1, \alpha \rangle [1, \alpha^2] = [1, \alpha^2] \perp [\alpha, \alpha]$  is isotropic, as both 2-dimensional forms represent  $\alpha^2$ , so the symbol is alternating.  $\square$

**Theorem 9.4.** *The function  $((\cdot, \dots, \cdot)) : F \times \cdots \times F \rightarrow I_q^n F$  is multi-additive and alternating.*

*Proof.* For  $n = 1$ , namely  $((\alpha)) = \langle \langle \alpha \rangle \rangle$ , there is nothing to prove. For  $n = 2$  we proved the relations in Proposition 9.3. So we may assume  $n > 2$ . The symbol is clearly symmetric in the first  $n - 1$  entries, and additive in the final entry. We will show the symbol is symmetric and multi-additive. For that,



it suffices to check for symmetry of the final two entries. Indeed, we have:

$$\begin{aligned}
& ((\alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta)) - ((\alpha_1, \dots, \alpha_{n-2}, \beta, \alpha_{n-1})) \\
&= \langle \langle \alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}, \alpha_1 \cdots \alpha_{n-1} \beta \rangle \rangle - \langle \langle \alpha_1, \dots, \alpha_{n-2}, \beta, \alpha_1 \cdots \alpha_{n-1} \beta \rangle \rangle \\
&= \langle \alpha_{n-1}, -\beta \rangle \langle \langle \alpha_1, \dots, \alpha_{n-2}, \alpha_1 \cdots \alpha_{n-1} \beta \rangle \rangle \\
&= \langle \langle \alpha_1, \dots, \alpha_{n-2} \rangle \rangle \langle \alpha_{n-1}, -\beta \rangle \langle \langle \alpha_1 \cdots \alpha_{n-1} \beta \rangle \rangle \\
&= \langle \langle \alpha_1, \dots, \alpha_{n-2} \rangle \rangle ([\alpha_{n-1}, \alpha_1 \cdots \alpha_{n-2} \beta] + [\beta, \alpha_1 \cdots \alpha_{n-1}]) \\
&= \langle \langle \alpha_1, \dots, \alpha_{n-2} \rangle \rangle ([\alpha_{n-1}, \alpha \beta] + [\beta, \alpha \alpha_{n-1}]) \\
&= \sum_{S \subseteq \{1, \dots, n-2\}} ([\alpha_S \alpha_{n-1}, \alpha_{S^c} \beta] + [\alpha_S \beta, \alpha_{S^c} \alpha_{n-1}]) \\
&= \sum [\alpha_S \alpha_{n-1}, \alpha_{S^c} \beta] + \sum [\alpha_{S^c} \beta, \alpha_S \alpha_{n-1}] = 0
\end{aligned}$$

It remains to prove that the form is alternating, and since  $n > 2$  it suffices to notice that  $\langle \langle \alpha, \alpha \rangle \rangle = 0$ , which is the case because  $\langle \alpha, \alpha \rangle \subseteq \langle \langle \alpha, \alpha \rangle \rangle$  and  $\langle \alpha, \alpha \rangle$  is isotropic.  $\square$

We now prove the first claim of Theorem 9.1. We take the forms

$$\psi_i = ((\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n)), \quad (3)$$

where  $i = 0, \dots, n$ . To show that  $\{\psi_0, \dots, \psi_n\}$  is strongly tight, we need to show that the sum of every subset of the generators  $\psi_0, \dots, \psi_{n+1}$  is a single Pfister form. Let  $S' \subseteq \{0, \dots, n\}$  be any subset. Permuting  $\alpha_0, \dots, \alpha_n$  if necessary, we may assume  $0 \in S'$ . Consider the  $n$ -fold form  $\psi' = ((\dots))$  whose entries are the  $\alpha_i$  for  $i \notin S$ , and the entries  $\alpha_0 + \alpha_i$  for  $i \in S$ ,  $i \neq 0$ . For example if  $n = 5$  and  $S = \{0, 1, 3\}$ , we consider  $((\alpha_2, \alpha_4, \alpha_0 + \alpha_1, \alpha_0 + \alpha_3))$ . By additivity and the fact that when  $\alpha_0$  appears twice the form cancels, we see that  $\psi' = \sum_{i \in S} \psi_i$ .

## 10. Valuations and Pfister forms

In this section we complete the proof of Theorem 9.1 by showing that the forms  $\psi_0, \dots, \psi_n$  of Equation (3) do not have a 2-dimensional common subform.

We begin with a useful lemma on monomial values. Let  $F$  be a field of characteristic 2, with a valuation  $\nu: F \rightarrow \Gamma$ , where  $\Gamma$  is a totally ordered abelian group. Let  $\Gamma^+ = \{\gamma \in \Gamma: \gamma > 0\}$  denote the positive semigroup.

Let  $e_0, \dots, e_{2^k-1}$  denote the standard basis of the vector space  $W = F^{(2^k)}$ . Consider the  $k$ -fold Pfister form  $\varphi = \langle\langle\beta_k, \dots, \beta_1\rangle\rangle$  defined on  $W$ , where  $\beta_1, \dots, \beta_k \in F^\times$ . Let  $D(\varphi) \subseteq \Gamma/2\Gamma$  denote the set of values under  $\nu$  of the elements  $\varphi(w) \in F^\times$ , ranging over  $w \in W$ .

**Lemma 10.1.** *Assume  $\nu(\beta_1) < 0$  and the images of  $\nu(\beta_1), \nu(\beta_2), \dots, \nu(\beta_k)$  in the quotient group  $\Gamma/2\Gamma$  are linearly independent (over  $\mathbb{F}_2$ ).*

*Then  $\varphi$  is anisotropic, and for every  $w = \sum_{i=0, \dots, 2^k-1} w_i e_i \in W$ ,  $\nu(\varphi(w)) = \nu(\varphi(w_i e_i))$  for some  $i$ .*

*Proof.* By induction. For  $k = 1$ ,  $\varphi(w_0 e_0 + w_1 e_1) = w_0^2 + w_0 w_1 + \beta_1 w_1^2$ . If  $w_0 w_1 = 0$  we are done, so let  $t = w_1^{-1} w_0$  and write  $\varphi(w_0 e_0 + w_1 e_1) = w_1^2(t^2 + t + \beta_1)$ . Let  $\delta = \nu(t)$ . If  $0 \leq \delta$  then clearly  $\nu(\beta_1) < 0 \leq \nu(t) \leq \nu(t^2)$ , so  $\beta_1$  is the monomial with least value in  $t^2 + t + \beta_1$  and  $\nu(\varphi(w_0 e_0 + w_1 e_1)) = \nu(\varphi(w_1 e_1))$ . Otherwise  $\delta < 0$ , so  $\nu(t^2) < \nu(t)$ , but  $\nu(\beta_1) \neq \nu(t^2) \in 2\Gamma$  and the minimum of  $\nu(t^2)$ ,  $\nu(t)$  and  $\nu(\beta_1)$  is obtained only once, so  $\nu(\varphi(w_0 e_0 + w_1 e_1)) = \nu(\varphi(w_1 e_1))$  or  $\nu(\varphi(w_0 e_0 + w_1 e_1)) = \nu(w_1^2 t^2) = \nu(\varphi(w_0 e_0))$ .

Assume the claim holds for  $\varphi_0 = \langle\langle\beta_{k-1}, \dots, \beta_1\rangle\rangle$  defined on  $W_0$ , where  $W = W_0 \oplus W_1$ , so  $\varphi = \varphi_0 \perp \beta_k \varphi_0$ . Assume  $\varphi(w) = \varphi_0(w_0) + \beta_k \varphi_0(w_1) \neq 0$  where  $w = w_0 + w_1$  and  $w_0 \in W_0$ ,  $w_1 \in W_1$ . If  $\varphi_0(w_0) = 0$  or  $\varphi(w_1) = 0$ , we are done; otherwise  $\nu(\varphi_0(w_0)) \neq \nu(\beta_k \varphi_0(w_1))$  by the induction hypothesis and the linear independence.  $\square$

**Corollary 10.2.** *Under the assumptions of lemma 10.1,  $D(\varphi)$  is the subspace of  $\Gamma/2\Gamma$  spanned by  $\nu(\beta_1), \dots, \nu(\beta_k)$ .*

**Corollary 10.3.** *Under the assumptions of lemma 10.1, for every 2-dimensional subspace  $U \subseteq W$ ,  $D(\varphi|_U) \neq 0$ .*

*Proof.* Since  $U$  is 2-dimensional, it contains a nonzero element  $w$  in which the coefficient of  $e_0$  is zero. Then by the previous lemma,  $\nu(\varphi(w)) = \nu(\varphi(w_t e_t)) = \nu(w_t^2) + \nu(e_t) \in \nu(e_t) + 2\Gamma$  for some  $t \neq 0$ . But  $\nu(e_t)$  is a nonempty partial sum of  $\nu(\beta_1), \dots, \nu(\beta_k)$  which are linearly independent, so it is not in  $2\Gamma$ .  $\square$

To complete the proof of Theorem 9.1, it remains to show that the  $n$ -fold Pfister forms  $\psi_i$  of Equation (3) do not share a common two-dimensional subform.

*Completion of the proof of Theorem 9.1.* We endow  $E = k(\alpha_0, \dots, \alpha_n)$  with the  $(\alpha_0^{-1}, \dots, \alpha_n^{-1})$ -adic valuation whose value set is  $\Gamma = \mathbb{Z}^{n+1}$ , with some

total ordering. Let  $\pi_i: \Gamma \rightarrow \mathbb{Z}/2\mathbb{Z}$  be the projection on the  $i$ th component, modulo 2. For each  $i$ , the form

$$\psi_i = \langle \alpha_0, \dots, \widehat{\alpha}_i, \dots, \alpha_{n-2}, \alpha_0 \cdots \widehat{\alpha}_i \cdots \alpha_{n-1} \rangle$$

satisfies the conditions of lemma 10.1, so by Corollary 10.2,  $D(\psi_i) = \ker(\pi_i)$ . Therefore,  $\bigcap D(\psi_i) = \bigcap \ker \pi_i = (2\mathbb{Z})^{n+1}$ . Let  $U \subseteq W$  be a two-dimensional subspace, and let  $\rho_i$  be the restriction of  $\psi_i$  to  $U$ . If we assume all the  $\rho_i$  are isometric to a form  $\rho$ , then  $D(\rho) \subseteq D(\psi_i)$  forces  $D(\rho) \subseteq (2\mathbb{Z})^{n+1}$ , which contradicts Corollary 10.3.  $\square$

This proves the second claim in Theorem 9.1.

## 11. Triplets of quaternion algebras

In this subsection we consider triplets of quaternion algebras. We show how the invariant can be used to prove the ‘additivity’ of left-linkage from the right-linkage of a triplet, and provide an example where the invariant of a tight triplet is nonzero in  $I_q^3 F / I_q^4 F$ .

### 11.1. Pairwise right-linked triplets

Let  $Q_i = [\alpha_i, \beta_i]$ ,  $i = 1, 2, 3$ , be three quaternion algebras over a field  $F$  of characteristic 2, and  $\varphi_i = \langle \langle \beta_i, \alpha_i \rangle \rangle$  the respective norm forms. In order for the invariant  $\Sigma_{\varphi_1, \varphi_2, \varphi_3}$  to be defined, the triplet must be tight. In this case, there are quaternion algebras  $Q_{ij}$  and  $Q_{123}$  such that  $Q_{ij} \sim Q_i \otimes Q_j$  and  $Q_{123} \sim Q_1 \otimes Q_2 \otimes Q_3$ . By definition the invariant is

$$\Sigma = \varphi_1 \perp \varphi_2 \perp \varphi_3 \perp \varphi_{12} \perp \varphi_{13} \perp \varphi_{23} \perp \varphi_{123}, \quad (4)$$

where each of the summands is the norm form of the respective algebra, namely a 2-fold Pfister form. It follows that  $\dim(\Sigma) = 28$ . By Proposition 8.4,  $\dim(\Sigma_{\text{an}}) \leq 16$ .

**Remark 11.1.** Assume  $\varphi_1, \varphi_2, \varphi_3$  are pairwise right-linked (without assuming tightness).

The algebras  $Q_{ij}$  are defined, but  $Q_{123}$  may not be. The triplet is tight if and only if  $\text{ind}(Q_1 \otimes Q_2 \otimes Q_3) \leq 2$ , or equivalently  $\text{ind}(Q_i \otimes Q_{jk}) \leq 2$  (where  $\{i, j, k\} = \{1, 2, 3\}$ ). This is equivalent to the Albert form  $\mathfrak{a}_i$  of  $Q_i \otimes Q_{jk}$  being isotropic (see Proposition 8.2).

**Lemma 11.2.** *Assume  $\varphi_1, \varphi_2, \varphi_3$  is a tight triplet of quaternion algebras. If the pairs  $\varphi_1, \varphi_2$  and  $\varphi_1, \varphi_3$  are left-linked, then  $\Sigma_{\varphi_1, \varphi_2, \varphi_3} = \Sigma_{\varphi_1, \varphi_{23}}$ .*

*Proof.* By assumption  $\varphi_1 + \varphi_2 = \varphi_{12}$  and  $\varphi_1 + \varphi_3 = \varphi_{13}$ , so by Equation (4) we have that  $\Sigma_{\varphi_1, \varphi_2, \varphi_3} = \varphi_1 \perp \varphi_{23} \perp \varphi_{123} = \Sigma_{\varphi_1, \varphi_{23}}$ .  $\square$

**Proposition 11.3.** *Assume  $\varphi_1, \varphi_2, \varphi_3$  is right-linked, and that the pairs  $\varphi_1, \varphi_2$  and  $\varphi_1, \varphi_3$  are left-linked. Then  $\varphi_1, \varphi_{23}$  are left-linked as well.*

*Proof.* By lemma 11.2 and Corollary 8.6,  $\Sigma_{\varphi_1, \varphi_{23}} = \Sigma_{\varphi_1, \varphi_2, \varphi_3} = 0$ , so the claim follows from Proposition 6.2.  $\square$

### 11.2. A triplet with nonzero invariant

We give an example of a tight triplet of 2-folds Pfister forms which cannot be right-linked because its invariant in  $I^3 E / I^4 E$  is nonzero.

**Proposition 11.4.** *Assume each of the pairs  $\varphi_1, \varphi_2$  and  $\varphi_1, \varphi_3$  is left-linked; the pair  $\varphi_2, \varphi_3$  is right-linked; but  $\varphi_1$  is not right-linked to  $\varphi_{23}$ .*

*Then, letting  $\mathfrak{a}$  be the Albert form of  $Q_1 \otimes Q_{23}$  (which is anisotropic by Proposition 8.2) and taking  $E = F(\mathfrak{a})$ , the set  $S = \{\varphi_1, \varphi_2, \varphi_3\}$  is tight over  $E = F(\mathfrak{a})$ , but  $\Sigma_S \notin I_q^4 E$ .*

*Proof.* We work over  $E$ , so the triplet is tight. By lemma 11.2,  $\Sigma_S = \Sigma_{\varphi_1, \varphi_{23}}$  so  $\dim(\Sigma_{\text{an}}) \leq 8$ . If  $\Sigma \in I_q^4 E$ , it follows that  $\Sigma = 0$ , which implies that  $\varphi_1, \varphi_{23}$  are left-linked over  $E$ , and by lemma 8.3 it follows that  $\varphi_1, \varphi_{23}$  are right-linked over  $F$ , contrary to assumption.  $\square$

We conclude with an example which realizes the conditions of Proposition 11.4. We need the following variant of lemma 10.1:

**Lemma 11.5.** *Let  $K$  be a field of characteristic 2 with a discrete valuation  $\nu: K \rightarrow \Gamma$ , and  $\alpha, \beta, \gamma \in K$ . Assume  $\nu(\beta) = \nu(\gamma) < 0$  and the images of  $\nu(\alpha), \nu(\beta)$  are linearly independent in  $\Gamma/2\Gamma$ . Then  $\alpha[1, \beta] \perp [1, \gamma]$  is anisotropic.*

*Proof.* Since  $\nu(\beta) = \nu(\gamma) \neq 0$  in  $\Gamma/2\Gamma$ , if  $d$  is a nonzero value of  $\langle\langle\beta\rangle\rangle$  or of  $\langle\langle\gamma\rangle\rangle$ , then  $\nu(d) \in \{0, \nu(\beta)\}$ , where we consider values modulo  $2\Gamma$ . Therefore, a nonzero value  $d$  of  $\alpha\langle\langle\beta\rangle\rangle$  has  $\nu(d) \in \{\nu(\alpha), \nu(\beta) + \nu(\alpha)\}$ , which is disjoint from the set of possible values for  $\langle\langle\gamma\rangle\rangle$ .  $\square$

**Example 11.6** (Realization of Proposition 11.4). *Let  $k$  a field of characteristic 2. Let  $F = k(\alpha, \beta, \gamma)$  be the function field in three algebraically independent variables over  $k$ . Let  $\phi_1 = \langle\langle\gamma, \alpha\rangle\rangle$ ,  $\phi_2 = \langle\langle\gamma, \beta\rangle\rangle$  and  $\phi_3 = \langle\langle\alpha\gamma, \beta\rangle\rangle$ . Clearly  $\phi_2$  is right-linked to  $\phi_3$  and  $\phi_1$  is left-linked to  $\phi_2$ . Since  $\phi_1 \cong \langle\langle\alpha\gamma, \alpha\rangle\rangle$ ,  $\phi_1$  is also left-linked to  $\phi_3$ . Let  $\phi_{23}$  denote the form  $\langle\langle\alpha, \beta\rangle\rangle$  which is equivalent modulo  $I_q^3 F$  to  $\phi_2 \perp \phi_3$ .*

*We need to show that  $\phi_1$  is not right-linked to  $\phi_{23}$ . By Proposition 8.2, we need to show that the Albert form  $\mathfrak{a} = \gamma[1, \alpha] \perp \alpha[1, \beta] \perp [1, \alpha + \beta]$  is anisotropic. Since  $F = k(\alpha, \beta)(\gamma)$ ,  $\mathfrak{a}$  is anisotropic if and only if  $[1, \alpha]$  and  $\alpha[1, \beta] \perp [1, \alpha + \beta]$  are anisotropic over  $k(\alpha, \beta)$ . But  $[1, \alpha]$  is clearly anisotropic, and  $\alpha[1, \beta] \perp [1, \alpha + \beta]$  is anisotropic by lemma 11.5 once we associate  $K = k(\alpha, \beta)$  with the  $(\alpha^{-1}, \beta^{-1})$ -valuation, with  $\nu(\alpha) > \nu(\beta)$ .*

## Acknowledgements

We thank Jean-Pierre Tignol and Adrian Wadsworth for their comments on the manuscript.

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